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Quadratic Spline Interpolation

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In this paper we study variational properties and convergence of quadratic spline interpolation where the points of interpolation are kept uniformly away from the mesh points.

1. INTRODUCTION

Marsden [1] showed that quadratic spline interpolation at the midpoints of mesh intervals gives rise to projections that are uniformly bounded in $C[0, 1]$. In [2], Kammerer *et al.* extended Marsden's result by proving, among other things, convergence of derivatives and a local convergence theorem. Demko [4] showed that, for the class of Lipschitz continuous functions, quadratic spline interpolation converges at the correct rate as long as the points of interpolation are kept uniformly away from the mesh points. In [5], Sharma and Tzimbarlario studied the variational properties of certain kinds of quadratic splines. Here we study quadratic spline interpolation methods where the points of interpolation are kept away from the mesh points. One of the purposes is to study variational properties that extend the results of [5]. Another purpose is to study the convergence of splines and their derivatives for the class of functions $C^r[a, b]$ $(r = 0, 1, 2, 3)$. We extend Demko's results and obtain error bounds for some smooth functions.

2. INTERPOLATION PROBLEMS

For $-\infty < a < b < +\infty$ and for any positive integer $n \geqslant 2$, let

$$\Delta_1: \qquad a = x_0 < x_1 < \cdots < x_n = b,$$

$$\Delta_2: \qquad a = y_0 < y_1 < \cdots < y_{n+1} = b$$

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denote two partitions of $[a, b]$, respectively, with knots x_i, y_i. The relationship between Δ_1 and Δ_2 is

$$y_0 = x_0 < y_1 < x_1 < \cdots < y_n < x_n = y_{n+1}.$$

Let $\mathrm{Sp}(\Delta_k, 2)$ denote the quadratic polynomial spline class defined on the partion Δ_k.

We have two types of interpolation problems.

1-*type interpolation problem*: Finding $s(x) \in \mathrm{Sp}(\Delta_1, 2)$ such that

$$s(y_j) = f_j \qquad (j = 0, 1, ..., n+1). \tag{2.1}$$

2–*type interpolation problem*: Finding $s(x) \in \mathrm{Sp}(\Delta_2, 2)$ such that

$$s(x_j) = f_j \qquad (j = 0, 1, ..., n), \tag{2.2}$$

$$s'(x_0) = f'_0, \qquad s'(x_n) = f'_n. \tag{2.3}$$

THEOREM 2.1. *The solution of the k-type* ($k = 1, 2$) *interpolation problem uniquely exists.*

Proof. The proofs for $k = 1$ and 2 are similar, so we just show the case $k = 1$.

It is enough to show the homogeneous interpolation problem has only the trivial solution.

By $s(x) \in C^1[a, b]$ and Rolle's theorem here must exist $z_i \in (y_i, y_{i+1})$ such that

$$s'(z_i) = 0 \qquad (i = 0, 1, ..., n).$$

Therefore the solution $s(x)$ must satisfy the following $2n + 1$ conditions:

$$s(y_i) = 0 \qquad (i = 0, 1, ..., n+1),$$

$$s'(z_i) = 0 \qquad (i = 1, 2, ..., n-1).$$

There are $2n + 1$ conditions, but only n subintervals, so there must exist a subinterval $[x_i, x_{i+1}]$ in which

$$s'(z_{i-1}) = 0, \qquad s(y_i) = 0, \qquad s'(z_i) = 0,$$

which uniquely define a quadratic polynomial $s(x) \equiv 0$ on the interval $[x_i, x_{i+1}]$. Then we obtain

$$s(x_i) = 0, \qquad s'(x_i) = 0, \qquad s(x_{i+1}) = 0, \qquad s'(x_{i+1}) = 0.$$

Therefore in the adjacent subintervals to $[x_i, x_{i+1}]$ there are three conditions which uniquely define $s(x) \equiv 0$ on these subintervals. Finitely repeating the process above we obtain $s(x) \equiv 0$ on the interval $[a, b]$. Q.E.D.

Remark. For the following interpolation problems whose end point interpolation conditions are changed the solution uniquely exists too. The problems are as follows.

I-type interpolation problem: Finding $s(x) \in \mathrm{Sp}(\Delta_1, 2)$ such that

$$\begin{aligned} s(y_j) &= f_j && (j = 1, 2, \ldots, n), \\ s^{(r)}(y_0 + 0) &= f_0^{(r)} && (r = 0, \text{ or } 1, \text{ or } 2), \\ s^{(r)}(y_{n+1} - 0) &= f_{n+1}^{(r)} && (r = 0, \text{ or } 1, \text{ or } 2). \end{aligned}$$

II-type interpolation problem: Finding $s(x) \in \mathrm{Sp}(\Delta_2, 2)$ such that

$$\begin{aligned} s(x_j) &= f_j && (j = 1, 2, \ldots, n-1), \\ s^{(r)}(x_0 + 0) &= f_0^{(r)} && (r = \text{any two among } 0, 1, 2), \\ s^{(r)}(x_n - 0) &= f_n^{(r)} && (r = \text{any two among } 0, 1, 2). \end{aligned}$$

3. Variational Properties

Let

$$PC^m[a, b] = \{ g(x) \mid g(x) \in C^{m-1}[a, b], g^{(m)}(x) \text{ is piecewice continuous on } [a, b] \text{ and there are only a finite number of discontinuous points of the first kind} \},$$

$$PC_1^m[a, b] = \{ g(x) \mid g(x) \in PC^m[a, b], g(y_j) = f_j, j = 0, 1, \ldots, n+1 \},$$

$$PC_2^m[a, b] = \{ g(x) \mid g(x) \in PC^m[a, b], g(x_j) = f_j, j = 0, 1, \ldots, n \}.$$

Set

$$u_1(t) = \frac{y_{j+1} - t}{y_{j+1} - y_j}, \qquad u_2(t) = \frac{t - y_j}{y_{j+1} - y_j},$$

$$v_1(t) = \frac{x_{j+1} - t}{x_{j+1} - x_j}, \qquad v_2(t) = \frac{t - x_j}{x_{j+1} - x_j},$$

$$\begin{aligned} p_1(t) &= y_j u_1(t) + x_j u_2(t), & p_2(t) &= x_j u_1(t) + y_j u_2(t), \\ p_3(t) &= x_j u_1(t) + y_{j+1} u_2(t), & p_4(t) &= y_{j+1} u_1(t) + x_j u_2(t), \\ q_1(t) &= x_j v_1(t) + y_{j+1} v_2(t), & q_2(t) &= y_{j+1} v_1(t) + x_j v_2(t), \\ q_3(t) &= y_{j+1} v_1(t) + x_{j+1} v_2(t), & q_4(t) &= x_{j+1} v_1(t) + y_{j+1} v_2(t). \end{aligned}$$

Consider two kinds of functional:

$$J_k[f] = \sum_{j=0}^{n+1-k} J_{kj}[f] \qquad (k = 1, 2),$$

where

$$J_{1j}[f] = \int_{y_j}^{y_{j+1}} \{[f'(p_1(t)) + f'(p_2(t))]u_2(x_j) + [f'(p_3(t)) + f'(p_4(t))]\, u_1(x_j)\}^2\, dt,$$

$$J_{2j}[f] = \int_{x_j}^{x_{j+1}} \{[f'(q_1(t)) + f'(q_2(t))]\, v_2(y_{j+1}) + [f'(q_3(t)) + f'(q_4(t))]\, v_1(y_{j+1})\}^2\, dt \qquad (j = 0, 1, ..., n+1-k),$$

$$f'(p_i(t)) = \frac{df(x)}{dx}\bigg|_{x=p_i(t)}, \qquad f'(q_i(t)) = \frac{df(x)}{dx}\bigg|_{x=q_i(t)} \qquad (i = 1, 2, 3, 4).$$

THEOREM 3.2. *Let $f(x) \in PC^2[a, b]$. Then $f(x)$ is a solution of the functional equation*

$$J_{kj}[f] = 0 \tag{3.1}$$

if and only if

$$k = 1: \qquad f(p_1(t)) - f(p_2(t)) + f(p_3(t)) - f(p_4(t)) = 0, \tag{3.2}$$

$$k = 2: \qquad f(q_1(t)) - f(q_2(t)) + f(q_3(t)) - f(q_4(t)) = 0. \tag{3.3}$$

Proof. Let $k = 1$. The sufficient condition is obvious. Let us show the necessary condition. By (3.1) we obtain

$$[f'(p_1(t)) + f'(p_2(t))]\, u_2(x_j) + [f'(p_3(t)) + f'(p_4(t))]\, u_1(x_j) = 0, \qquad t \in [y_j, y_j + 1].$$

Integrating,

$$f(p_1(t)) - f(p_2(t)) + f(p_3(t)) - f(p_4(t)) = c.$$

Setting

$$t = \tfrac{1}{2}(y_j + y_{j+1}),$$

we obtain

$$f\left(\frac{x_j + y_j}{2}\right) - f\left(\frac{x_j + y_j}{2}\right) + f\left(\frac{x_j + y_{j+1}}{2}\right) - f\left(\frac{x_j + y_{j+1}}{2}\right) = c,$$

i.e. $c = 0$. Therefore (3.2) is valid.

For $k = 2$ the proof is similar. Q.E.D.

Corollary 3.2.1. *Let $f(x) \in PC^2[a,b]$. If $f(x)$ is a solution of the functional equation* (3.1) *then*

$$k=1:\quad f(y_j)=f(y_{j+1}),$$
$$k=2:\quad f(x_j)=f(x_{j+1}).$$

Corollary 3.2.2. *Let $f(x) \in PC^2[a,b]$. If*

$k=1$: *on $[y_j, x_j], [x_j, y_{j+1}]$ $f(x)$ is symmetrical, respectively, about the midpoint of these intervals,*

$k \mp 2$: *on $[x_j, y_{j+1}], [y_{j+1}, x_{j+1}]$ $f(x)$ is symmetrical, respectively, about the midpoint of these intervals,*

then $f(x)$ is a solution of the functional equation (3.1).

By the theorem above we directly obtain the following theorem.

Theorem 3.3. *Let $f(x) \in PC^2[a,b]$. Then $f(x)$ is a solution of the functional equation $J_k[f]=0$ $(k=1,2)$ if and only if*

$$k=1:\quad f(p_1(t))-f(p_2(t))+f(p_3(t))-f(p_4(t))=0,$$
$$t \in [y_j, y_{j+1}],\qquad j=0,1,\ldots,n;$$
$$k=2:\quad f(q_1(t))-f(q_2(t))+f(q_3(t))-f(q_4(t))=0,$$
$$t \in [x_j, x_{j+1}],\quad j=0,1,\ldots,n-1.$$

Corollary 3.3.1. *Let $f(x) \in PC^2[a,b]$. If $f(x)$ is a solution of the functional equation $J_k[f]=0$ $(k=1,2)$ then*

$$k=1:\quad f(y_0)=f(y_1)=\cdots=f(y_{n+1}),$$
$$k=2:\quad f(x_0)=f(x_1)=\cdots=f(x_n).$$

Corollary 3.3.2. *Let $f(x) \in PC^2[a,b]$. If on the intervals $[x_j, y_{j+1}]$, $[y_{j+1}, x_{j+1}]$ $(j=0,1,\ldots,n-1)$ $f(x)$ is symmetrical, respectively, about the midpoint of these intervals then $f(x)$ is a solution of the functional equation $J_k[f]=0$.*

If $f(x) \in \mathrm{Sp}(\Delta_k, 2)$ the sufficient and necessary conditions will be simplified.

Theorem 3.4. *Let $s(x) \in \mathrm{Sp}(\Delta_k, 2)$. Then $s(x)$ is a solution of* (3.1) *if and only if*

$$k=1:\quad s(y_j)=s(y_{j+1}),$$
$$k=2:\quad s(x_j)=s(x_{j+1}).$$

Proof. The necessary conditions have been given by Corollary 3.2.1. Let $k=1$. Integrating by parts,

$$\begin{aligned} J_{1j}[s] = {} & [s(x_j) - s(y_j) + s(y_{j+1}) - s(x_j)][(s'(x_j) + s'(y_j))\, u_2(x_j) \\ & + (s'(y_{j+1}) + s'(x_j))\, u_1(x_j)] - [s(y_j) - s(x_j) + s(x_j) - s(y_{j+1})] \\ & \times [(s'(y_j) + s'(x_j))\, u_2(x_j) + (s'(x_j) + s'(y_{j+1}))\, u_1(x_j)] \\ & - \int_{y_j}^{y_{j+1}} [s(p_1(t)) - s(p_2(t)) + s(p_3(t)) - s(p_4(t))][(s''(x_j - 0) \\ & - s''(x_j - 0))\, u_2^2(x_j) - (s''(x_j + 0) - s''(x_j + 0))\, u_1^2(x_j)]\, dt = 0. \end{aligned}$$

For $k=2$ the proof is similar. Q.E.D.

THEOREM 3.5. *Let* $s(x) \in \mathrm{Sp}(\Delta_k, 2)$. *Then* $s(x)$ *is a solution of the functional equation* $J_k[s] = 0$ *if and only if*

$$k=1: \quad s(y_0) = s(y_1) = \cdots = s(y_{n+1}),$$

$$k=2: \quad s(x_0) = s(x_1) = \cdots = s(x_n).$$

This theorem is directly obtained from Theorem 3.4.

COROLLARY 3.5.1. *Let* $s(x) \in \mathrm{Sp}(\Delta_1, 2)$. *Then* $s(x)$ *is a solution of* $J_1[s] = 0$ *if and only if* $s(x) = \text{const}$.

COROLLARY 3.5.2. *Let* $s(x) \in \mathrm{Sp}(\Delta_2, 2)$ *and* $s'(x_0) = s'(x_n) = 0$. *Then* $s(x)$ *is a solution of* $J_2[s] = 0$ *if and only if* $s(x) = \text{const}$.

THEOREM 3.6. *Let* $s(x) \in \mathrm{Sp}(\Delta_k, 2)$ *be the solution of a k-type* $(k = 1, 2)$ *interpolation problem with* $f(x) \in PC_k^2[a, b]$. *Then there exist the first integration relationships*:

$$\textit{local}: \quad J_{kj}[f] = J_{kj}[s] + J_{kj}[f - s] \qquad (j = 0, 1, \ldots, n+1-k), \tag{3.4}$$

$$\textit{global}: \quad J_k[f] = J_k[s] + J_k[f - s] \qquad (k = 1, 2). \tag{3.5}$$

Proof. Let $k=1$. We have

$$J_{1j}[f - s] = J_{1j}[f] - J_{1j}[s] - 2I[f - s, s],$$

where

$$
\begin{aligned}
I[f-s, s] = \int_{y_j}^{y_{j+1}} & \{[f'(p_1(t)) - s'(p_1(t)) \\
& + f'(p_2(t)) - s'(p_2(t))] \, u_2(x_j) + [f'(p_3(t)) - s'(p_3(t)) \\
& + f'(p_4(t)) - s'(p_4(t))] \, u_1(x_j)\} \{[s'(p_1(t)) \\
& + s'(p_2(t))] \, u_2(x_j) + [s'(p_3(t)) + s'(p_4(t))] \, u_1(x_j)\} \, dt. \qquad (3.6)
\end{aligned}
$$

Integrating $I[f-s, s]$, by parts, we obtain

$$
\begin{aligned}
I[f-s, s] = & [(f(x_j) - s(x_j)) - (f(y_j) - s(y_j)) \\
& + (f(y_{j+1}) - s(y_{j+1})) - (f(x_j) - s(x_j))][(s'(x_j) \\
& + s'(y_j)) \, u_2(x_j) + (s'(y_{j+1}) + s'(x_j)) \, u_1(x_j)] \\
& - [(f(y_j) - s(y_j)) - (f(x_j) - s(x_j)) + (f(x_j) - s(x_j)) \\
& - (f(y_{j+1}) - s(y_{j+1}))][(s'(y_j) + s'(x_j)) \, u_2(x_j) \\
& + (s'(x_j) + s'(y_{j+1})) \, u_1(x_j)] \\
& - \int_{y_j}^{y_{j+1}} \{[f(p_1(t)) - s(p_1(t))] - [(f(p_2(t)) - s(p_2(t))] \\
& + [f(p_3(t)) - s(p_3(t))] - [f(p_4(t)) - s(p_4(t))]\} \\
& \times [(s''(x_j - 0) - s''(x_j - 0))] \, u_2(x_j) \\
& - (s''(x_j + 0) - s''(x_j + 0)) \, u_1(x_j)] \, dt = 0,
\end{aligned}
$$

so (3.4) and (3.5) are valid for $k = 1$. For $k = 2$ the proof is similar.

Q.E.D.

THEOREM 3.7. *With $f(x)$ and $s(x)$ as in Theorem 3.6, there must be*

$$
\begin{aligned}
&\textit{local}: \quad J_{kj}[s] \leqslant J_{kj}[f] \qquad (j = 0, 1, \ldots, n + 1 - k), \\
&\textit{global}: \quad J_k[s] \leqslant J_k[f] \qquad (k = 1, 2).
\end{aligned}
$$

Proof. Because $J_{kj}[f-s] \geqslant 0$, $J_k[f-s] \geqslant 0$, by the first integration relationship we directly obtain Theorem 3.7.

Remark. By Corollary 3.3.2 we know that $J_k[f]$ does not uniquely attain a smallest value in the class of $PC_k^2[a, b]$ functions which interpolate a given k-type data set. However, by the corollaries to Theorem 3.5 we know that uniqueness does hold in the class $\mathrm{Sp}(\Delta_k, 2)$.

THEOREM 3.8. *Let $f(x) \in PC_k^2[a,b]$, and let $s_f(x) \in \mathrm{Sp}(\Delta_k, 2)$ be the solution of the k-type $(k=1,2)$ interpolation problem for $f(x)$ and $s(x) \in \mathrm{Sp}(\Delta_k, 2)$ be an arbitrary quadratic spline function. Then there are*

$$\textit{local}: \quad J_{kj}[f-s_f] \leqslant J_{kj}[f-s[\,] \qquad (j=0,1,\ldots,n+1-k), \tag{3.7}$$

$$\textit{global}: \quad J_k[f-s_f] \leqslant J_k[f-s] \qquad (k=1,2). \tag{3.8}$$

Proof. We have

$$J_{kj}[f-s] = J_{kj}[f-s_f] + J_{kj}[s_f-s] + 2I[f-s_f, s_f-s],$$

where $I[f-s_f, s_f-s]$ is similar to (3.6) and easily shown to be zero. Therefore

$$J_{kj}[f-s] = J_{kj}[s_f-s] + J_{kj}[f-s_f].$$

But

$$J_{kj}[s_f-s] \geqslant 0,$$

so (3.7) and (3.8) are valid.

Remark. By the corollaries of Theorem 3.5 we know that if $s(x)$ and $s_f(x)$ have the same end point values the equality case of (3.8) holds only if $s(x) \equiv s_f(x)$.

THEOREM 3.9. *Let $f(x) \in PC_k^2[a,b]$, and let $s_f(x) \in \mathrm{Sp}(\Delta_k, 2)$ be the solution of the k-type $(k=1,2)$ interpolation problem for $f(x)$. Then there are the second integration relationships*

$$\textit{local}: \quad J_{kj}[\; - s_f = -L_{kj}[f-s_f, f] \qquad (j=0,1,\ldots,n+1-k),$$

$$\textit{global}: \quad J_k[f-s_f] = -L_k[f-s_f, f] \qquad (k=1,2),$$

where

$$\begin{aligned} L_{1j}[f-s_f, f] = \int_{y_j}^{y_{j+1}} &\{[f(p_1(t)) - s_f(p_1(t))] - [f(p_2(t)) - s_f(p_2(t))] \\ &+ [f(p_3(t)) - s_f(p_3(t))] - [f(p_4(t)) - s_f(p_4(t))]\} \\ &\times \{[f''(p_1(t)) - f''(p_2(t))]\, u_2^2(x_j) \\ &+ [f''(p_3(t)) - f''(p_4(t))]\, u_1^2(x_j)\}\, dt, \end{aligned}$$

$$L_{2j}[f-s_f,f]=\int_{x_j}^{x_{j+1}}\{[f(q_1(t))-s_f(q_1(t))]-[f(q_2(t))-s_f(q_2(t))]$$
$$+[f(q_3(t))-s_f(q_3(t))]-[f(q_4(t))-s_f(q_4(t))]\}$$
$$\times\{[f''(q_1(t))-f''(q_2(t))]\,v_2^2(y_{j+1})$$
$$+[f''(q_3(t))-f''(q_4(t))]\,v_1^2(y_{j+1})\}\,dt,$$

$$L_k[f-s_f,f]=\sum_{j=0}^{n+1-k}L_{kj}[f-s_f,f].$$

Proof. By integration by parts we obtain this theorem.

4. Convergence

When we discuss the 1-type interpolation problem we use the following symbols:

$$h_i=y_{i+1}-y_i,\qquad t_i=y_{i+1}-x_i,\qquad i=0,1,\ldots,n,$$
$$m_i=s'(y_i),\qquad M_i=s''(y_i),\qquad i=0,1,\ldots,n+1.$$

The solution of the 1-type interpolation problem can be expressed as follows:

$$s(x)=f_i+m_i(x-y_i)+M_i(x-y_i)^2/2+d_i(x-x_i)_+^2$$
$$(x\in[y_i,y_{i+1}],\ i=0,1,\ldots,n). \tag{4.1}$$

The parameters $m_1, m_2,\ldots, m_n$ satisfy the system of linear equations

$$a_{i+1}m_i+(2+a_{i+1}+b_{i+1})\,m_{i+1}+b_{i+1}m_{i+2}=r_{i+1}\qquad(i=0,1,\ldots,n-1), \tag{4.2}$$

where

$$a_{i-1}=\frac{h_{i+1}(h_i-t_i)}{t_i(h_i+h_{i+1})},\qquad b_{i+1}=\frac{t_{i+1}h_i}{(h_{i+1}-t_{i+1})(h_i+h_{i+1})} \tag{4.3}$$

$$r_{i+1}=\frac{2h_{i+1}(f_{i+1}-f_i)}{t_i(h_i+h_{i+1})}+\frac{2h_i(f_{i+2}-f_{i+1})}{(h_i+h_{i+1})(h_{i+1}-t_{i+1})}. \tag{4.4}$$

Also,

$$M_i = \frac{2(f_{i+1} - f_i)}{h_i(h_i - t_i)} - \frac{t_i m_{i+1}}{h_i(h_i - t_i)} - \frac{(2h_i - t_i)\, m_i}{h_i(h_i - t_i)} \qquad (i = 1, 2, \ldots, n), \tag{4.5}$$

$$M_i + 2d_i = \frac{t_i - h_i}{t_i h_i} m_{i-1} + \frac{h_i - t_i}{t_i h_i} m_i - \frac{2}{t_i h_i}(f_{i+1} - f_i) \qquad (i = 0, 1, \ldots, n-1). \tag{4.6}$$

The parameters $M_1, M_2, \ldots, M_n$ satisfy the system of linear equations

$$A_{i+1} M_i + (1 - A_{i+1} - B_{i+1})\, M_{i+1} + B_{i+1} M_{i+2} = R_{i+1} \qquad (i = 0, 1, \ldots, n-1), \tag{4.7}$$

where

$$A_{i+1} = \frac{(t_i - h_i)^2}{h_i(h_i + h_{i+1})}, \qquad B_{i+1} = \frac{t_{i+1}^2}{h_{i+1}(h_i + h_{i+1})}, \tag{4.8}$$

$$R_{i+1} = \frac{2}{h_i + h_{i+1}} \left(\frac{f_{i+2} - f_{i+1}}{h_{i+1}} - \frac{f_{i+1} - f_i}{h_i} \right). \tag{4.9}$$

Also,

$$m_i = (f_{i+1} - f_i)/h_i - M_i h_i/2 - (M_{i+1} - M_i)\, t_i^2/(2h_i) \qquad (i = 0, 1, \ldots, n), \tag{4.10}$$

$$d_i = (M_{i+1} - M_i)/2 \qquad (i = 0, 1, \ldots, n). \tag{4.11}$$

When we discuss the 2-type interpolation problem we use the following symbols:

$$h_i = x_{i+1} - x_i, \qquad t_i = x_{i+1} - y_{i+1}, \qquad i = 0, 1, \ldots, n-1,$$

$$m_i = s'(x_i), \qquad M_i = s''(x_i), \qquad i = 0, 1, \ldots, n.$$

The solution of the 2-type interpolation problem can be expressed as follows:

$$s(x) = f_i + m_i(x - x_i) + M_i(x - x_i)^2/2 + d_i(x - y_{i+1})_+^2, \qquad (x \in [x_i, x_{i+1}],\ i = 0, 1, \ldots, n-1). \tag{4.12}$$

The parameters $m_1, m_2, \ldots, m_{n-1}$ satisfy the system of linear equations

$$\begin{cases} (2 + a_1 + b_1)\, m_1 + b_1 m_2 = r_1 - a_1 f'_0, \\ a_{i+1} m_i + (2 + a_{i+1} + b_{i+1})\, m_{i+1} + b_{i+1} m_{i+2} = r_{i+1}, \quad (i = 1, 2, \ldots, n-3), \\ a_{n-1} m_{n-2} + (2 + a_{n-1} + b_{n-1})\, m_{n-1} = r_{n-1} - b_{n-1} f'_n, \end{cases} \tag{4.13}$$

where a_i, b_i, r_i are as in (4.3), (4.4) and M_i, d_i are as in (4.5), (4.6).

The parameters $M_0, M_1, \ldots, M_n$ satisfy the system of linear equations

$$\begin{cases} (1-A_0)M_0 + B_0 M_1 = 2((f_1 - f_0)/h_0 - f'_0)/h_0, \\ A_{i+1}M_i + (1-A_{i+1} - B_{i+1})M_{i+1} + B_{i+1}M_{i+2} = R_{i+1}, \quad (i = 0, 1, \ldots, n-2), \\ A_n M_{n-1} + (1-A_n)M_n = 2(f' - (f_n - f_{n-1})/h_{n-1})/h_{n-1}, \end{cases} \tag{4.14}$$

where A_i, B_i, R_i are as in (4.8) (4.9) and m_i, d_i are as in (4.10), (4.11).

For the sake of convenience set

$$\|g_i\| = \max_i |g_i|, \qquad \|g\| = \sup_{x \in [a,b]} |g(x)|,$$

$$\omega(g, h) = \sup_{\substack{x', x'' \in [a,b] \\ |x' - x''| \leqslant h}} |g(x') - g(x'')|,$$

$$h = \max_{0 \leqslant i \leqslant n+1-k} h_i,$$

$$\max_{0 \leqslant i \leqslant n+1-k} (t_i/h, (h_i - t_i)/h) \leqslant \alpha = \text{const},$$

$$e(x) \equiv f(x) - s(x).$$

The following lemma is easily shown.

LEMMA. *Let*

$$a_{i+1}u_i + b_{i+1}u_{i+1} + c_{i+1}u_{i+2} = r_{i+1} \qquad (i = 0, 1, \ldots, n-1).$$

If the coefficients a_i, b_i, c_i satisfy the conditions

$$\begin{aligned} &a_1 = c_n = 0, \qquad a_i \geqslant 0, \qquad c_i \geqslant 0, \qquad i = 1, 2, \ldots, n, \\ &b_i - (a_i + c_i) \geqslant K^{-1} \qquad (i = 1, 2, \ldots, n; K = \text{const}), \end{aligned} \tag{4.15}$$

then

$$\|u_i\| \leqslant K \|r_i\|.$$

THEOREM 4.10. *If $f(x) \in C^0[a, b]$ and partitions Δ_1, Δ_2 satisfy*

$$\max_{0 \leqslant i \leqslant n-k} (h/t_i, h/(h_{i-1} - t_{i-1})) \leqslant \beta = \text{const}. \tag{4.16}$$

then for the solution of the 1*-type interpolation problem*

$$\|e(x)\| \leqslant 2(1 + \alpha\beta)\, \omega(f, h). \tag{4.17}$$

Proof. Let $k=1$. Apply the lemma to system (4.2) to obtain

$$\|hr_{i-1}\| \leqslant 2\beta\omega(f, h),$$
$$h\,\|m_i\| \leqslant \beta\omega(f, h).$$

Estimating (4.5), (4.6), obtain

$$(h_i - t_i)^2\,|M_i| \leqslant 2(1+\alpha\beta)\,\omega(f, h) \qquad (i = 1, 2, ..., n),$$
$$t_i^2\,|M_i + 2d_i| \leqslant 2(1+\alpha\beta)\,\omega(f, h) \qquad (i = 0, 1, ..., n-1).$$

If $x \in [y_i, x_i]$ $(i = 1, 2, ..., n)$ then

$$s(x) = f_i + m_i(x - y_i) + M_i(x - y_i)^2/2,$$
$$|e(x)| \leqslant |f(x) - f_i| + (h_i - t_i)\,|m_i| + (h_i - t_i)^2\,|M_i|/2$$
$$\leqslant 2(1+\alpha\beta)\,\omega(f, h).$$

If $x \in [x_i, y_{i+1}]$ $(i = 0, 1, ..., n-1)$ then

$$s(x) = f_{i+1} + m_{i+1}(x - y_{i+1}) + (M_i + 2d_i)(x - y_{i+1})^2/2,$$
$$|e(x)| \leqslant 2(1+\alpha\beta)\,\omega(f, h).$$

For $k=2$ the proof is similar. Q.E.D.

When $f(x) \in C^0[a, b]$ $f'(x)$ may not be defined at $x=a$, $x=b$, so we change the end point conditions (2.3) of the 2-type interpolation problem to

$$s'(x_0) = (f_1 - f_0)/h_0, \qquad s'(x_n) = (f_n - f_{n-1})/h_{n-1}. \tag{4.18}$$

In this case we obtain a similar theorem.

THEOREM 4.11. *If $f(x) \in C^0[a, b]$ and partitions Δ_1, Δ_2 satisfy* (4.16), *then for the solution of the* 2*-type interpolation problem* (2.2), (4.18) *the estimation* (4.17) *holds.*

THEOREM 4.12. *If $f(x) \in C^1[a, b]$ and partitions Δ_1, Δ_2 satisfy*

$$\max_{0 \leqslant i \leqslant n-k} (h_i/t_i, h_{i+1}/(h_{i+1} - t_{i+1})) \leqslant \gamma_1 = \text{const}, \tag{4.19}$$

then for the solution and its derivative of the k-type $(k = 1, 2)$ *interpolation problem*

$$\|e'(x)\| \leqslant 3(1+\gamma_1)\,\omega(f', h),$$
$$\|e(x)\| \leqslant 3\alpha(1+\gamma_1)\,h\omega(f', h).$$

Proof. Let $k = 1$. Rewrite system (4.2) to obtain

$$a_{i+1}(m_i - f'_i) + (2 + a_{i+1} + b_{i+1})(m_{i+1} - f'_{i+1}) + b_{i+1}(m_{i+2} - f'_{i+2}) = \tilde{r}_{i+1},$$
$$\tilde{r}_{i+1} = r_{i+1} - a_{i+1} f'_i - (2 + a_{i+1} + b_{i+1}) f'_{i+1} - b_{i+1} f'_{i+2},$$
$$i = 0, 1, \ldots, n-1.$$

Apply the lemma to the system above to obtain

$$\|e'(x)\| \leqslant 3(1 + \gamma_1)\, \omega(f', h).$$

If $x \in [y_i, x_i]$ then

$$|e(x)| = \left| \int_{y_i}^{x} (f'(t) - s'(t))\, dt \right| \leqslant 3(1 + \gamma_1)\, \alpha h \omega(f', h).$$

If $x \in [x_i, y_{i+1}]$ then

$$|e(x)| = \left| \int_{y_{i+1}}^{x} (f'(t) - s'(t))\, dt \right| \leqslant 3(1 + \gamma_1)\, \alpha h \omega(f', h).$$

For $k = 2$ the proof is similar. Q.E.D.

COROLLARY 4.12. *If $f(x) \in C^2[a, b]$ and partitions Δ_1, Δ_2 satisfy* (4.19) *then for the solution and its derivative of the k-type* ($k = 1, 2$) *interpolation problem*

$$\|e'(x)\| \leqslant \tfrac{3}{4} \gamma_1 h \omega(f', h) + 3\, \|f''\|\, h,$$
$$\|e(x)\| \leqslant \tfrac{3}{4} \alpha \gamma_1 h^2 \omega(f', h) + 3\, \|f''\|\, h^2.$$

THEOREM 4.13. *If $f(x) \in C^2[a, b]$ and partitions Δ_1, Δ_2 satisfy*

$$h_i / t_i = \gamma = \text{const} \qquad (i = 2 - k, 3 - k, \ldots, n - 1), \tag{4.20}$$

and

$$\sqrt{2} < \gamma < 2 + \sqrt{2}, \tag{4.21}$$

then for the solution and its derivatives of the k-type ($k = 1, 2$) *interpolation problem*

$$\|e''(x)\| \leqslant (1 + c_1)\, \omega(f'', h),$$
$$\|e'(x)\| \leqslant (1 + c_1)\, h \omega(f'', h),$$
$$\|e(x)\| \leqslant \alpha(1 + c_1)\, h^2 \omega(f'', h),$$

where

$$c_1 = \begin{cases} (\gamma^2+1)/(\gamma^2-2) & (\textit{when } \sqrt{2} < \gamma \leqslant 2), \\ (2\gamma^2-2\gamma+1)/(4\gamma-\gamma^2-2) & (\textit{when } 2 \leqslant \gamma < 2+\sqrt{2}). \end{cases}$$

Applying the lemma to systems (4.7), (4.14) *obtains*

$$\|M_i - f_i''\| \leqslant c_1 \omega(f'', h).$$

The proof is finished by a demonstration similar to Theorem 4.12.

COROLLARY 4.13. *If* $f(x) \in C^3[a, b]$ *and partitions* Δ_1, Δ_2 *satisfy* (4.20), (4.21) *then for the solution and its derivatives of the k-type* ($k = 1, 2$) *interpolation problem*

$$\|e''(x)\| \leqslant (1+c_1)\, h\, \|f'''\|,$$
$$\|e'(x)\| \leqslant (1+c_1)\, h^2\, \|f'''\|,$$
$$\|e(x)\| \leqslant \alpha(1+c_1)\, h^3\, \|f'''\|.$$

THEOREM 4.14. *If* $f(x) \in C^2[a, b]$ *and partitions* Δ_1, Δ_2 *satisfy* (4.20) *and when* $\gamma \geqslant 2$ *the partitions* Δ_{3-k} *is nondecreasing, i.e.,*

$$h_{2-k} \leqslant h_{3-k} \leqslant \cdots \leqslant h_{n-1},$$

when $\gamma \leqslant 2$ *the partition* Δ_{3-k} *is nonincreasing, i.e.,*

$$h_{2-k} \geqslant h_{3-k} \geqslant \cdots \geqslant h_{n-1},$$

then for the solution and its derivatives of the k-type ($k = 1, 2$) *interpolation problem*

$$\|e''(x)\| \leqslant (1+c_2)\, \omega(f'', h),$$
$$\|e'(x)\| \leqslant (1+c_2)\, h\omega(f'', h),$$
$$\|e(x)\| \leqslant \alpha(1+c_2)\, h^2\omega(f'', h),$$

where $c_2 = (3\gamma^2 - 2\gamma + 2)/(4\gamma - 4)$.

THEOREM 4.15. *If* $f(x) \in C^2[a, b]$ *and partitions* Δ_1, Δ_2 *satisfy* (4.20) *and $\gamma = 2$ (i.e., midpoint), then for the solution and its derivatives of the k-type ($k = 1, 2$) interpolation problem*

$$\|e''(x)\| \leqslant 3.5\omega(f'', h),$$
$$\|e'(x)\| \leqslant 3.5h\omega(f'', h),$$
$$\|e(x)\| \leqslant 1.75h^2\omega(f'', h).$$

In fact, this theorem is a corollary of Theorem 4.13.

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